ON LOCALIZATION AND DOMAINS OF UNIQUENESS

BY
ROE GOODMAN(1)

1. **Introduction.** In this paper we consider the support properties of distributions $u(t)$, $t \in R^1$, on an open subset M of R^N which satisfy an abstract hyperbolic equation $du/dt = iAu$. Here $u(t)$ is assumed to be "normalizable," i.e., to belong to a Hilbert space H of (vector-valued) distributions, and A is a self-adjoint operator on H. Our general result is that if A has no homogeneous Lebesgue spectrum (see §3 for definition), then any restriction on the support of $u(t)$ for $t<0$ holds for all t (Theorem 3.1), so that if the support of $u(t)$ decreases to the empty set as $t \to -\infty$, then $u=0$ (Theorem 3.2). These results generalize and sharpen theorems in [1] (some of which were also proved by a different method using energy inequalities in [4]), where a more restricted class of equations was considered and stronger assumptions on the spectrum of A were made.

In §2 we make precise the classes of equations and spaces of distributions considered. §3 contains the main theorems on localization and domains of uniqueness, with applications in §§4 and 5. In §6 we discuss the converse of Theorems 3.1 and 3.2, and give a counterexample.

2. **Equations of evolution.** Let M be an open subset of R^N, V a finite-dimensional complex vector space, and D the space of C^∞ functions from M to V with compact support. Give D the usual locally convex inductive limit topology [5]. Suppose H is a Hilbert space embedded in D', i.e., a bilinear form $\langle \cdot, \cdot \rangle$ on $H \times D$ is assumed given, such that

(A) $\phi \to \langle f, \phi \rangle$ is continuous on D for each $f \in H$.

(B) $f \to \langle f, \phi \rangle$ is continuous on H for each $\phi \in D$.

(C) $\langle f, \phi \rangle = 0$ for all $\phi \in D$ implies $f=0$.

LEMMA 2.1. *For every compact set $K \subseteq M$ there exists an integer k and a constant C such that*

$$|\langle f, \phi \rangle| \leqq C \|f\|_H \|\phi\|_k$$

for all $f \in H$ and $\phi \in D_K$.

REMARK ON NOTATION. D_K consists of the functions in D supported on K, $\|\cdot\|_H$ is the Hilbert norm on H, and $\|\phi\|_k = \sup_{|\alpha| \leqq k} \|D^\alpha \phi\|_\infty$, where α is a multi-index, $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$, $D_j = \partial / i\, \partial x_j$, and $\|\cdot\|_\infty$ is the supremum norm.

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Proof of lemma. D_K and H are Fréchet spaces, and conditions (A) and (B) state that the bilinear form $\langle \cdot, \cdot \rangle$ restricted to $D_K \times H$ is separately continuous. Hence by the Banach-Steinhaus theorem it is jointly continuous, from which the lemma follows.

REMARK. In particular, H must be separable. Indeed, by conditions (B) and (C), the bilinear form defines a map $\pi: D \to H'$ with $\pi(D)$ total. Hence by convexity, $\pi(D)$ is norm-dense in H'. By Lemma 2.1, if $\phi \in D_K$ then $\|\pi(\phi)\|_{H'} \leqq C\|\phi\|_k$, from which the separability of H' and hence H follows.

Suppose that A is a self-adjoint operator on H, with domain $D[A]$, and let $W(t)$ be the one-parameter unitary group generated by A. Consider the abstract Cauchy problem

$$du/dt = iAu, \quad u(0) = f \in H. \tag{2.1}$$

Now $u(t) = W(t)f$ is defined for any $f \in H$, and by Stone's theorem is strongly differentiable and satisfies (2.1) in case $f \in D[A]$. We shall persist in calling $u(t)$ a *solution* of (2.1) however, for f arbitrary in H.

Let u be a solution of (2.1), in the above sense. For each t, $u(t)$ defines via the pairing $\langle \cdot, \cdot \rangle$ a V'-valued distribution on M. We now show that these distributions may be integrated with respect to t, so that u defines an element of D_1', where $D_1 = C^\infty$ functions from $M \times R^1$ to V with compact support. Denote the points of $M_1 = M \times R^1$ by (x, t), and for $\phi \in D_1$, $t \in R^1$, let $\phi(t) = \phi(\cdot, t) \in D$.

LEMMA 2.2. *For any $f \in H$ and $\phi \in D_1$, the function $t \to \langle W(t)f, \phi(t) \rangle$ is continuous with compact support, and the bilinear form*

$$\langle f, \phi \rangle_1 = \int \langle W(t)f, \phi(t) \rangle \, dt$$

satisfies conditions (A), (B), *and* (C).

Proof. For $\phi \in D_1$, the set $\{\phi(t) \mid t \in R^1\}$ is obviously bounded in D (i.e., is contained in D_K for some compact K and has bounded $\|\cdot\|_k$ norms for all k). Also W unitary implies $\{W(t)f \mid t \in R^1\}$ is bounded in H. It follows from Lemma 2.1 that $t \to \langle W(t)f, \phi(t) \rangle$ is continuous, and obviously has compact support. Furthermore, if $K_1 \subseteq R^1 \times M$ is compact, then there exists an integer k_1 and a constant C_1 such that

$$\langle f, \phi \rangle_1 \leqq C_1 \|f\|_H \|\phi\|_{k_1}$$

for all $f \in H$ and $\phi \in D_1$, supp $(\phi) \subseteq K_1$. This shows that the form $\langle \cdot, \cdot \rangle_1$ satisfies conditions (A) and (B). To verify condition (C), take $\phi \in D_1$ of the form $\phi(x, t) = \Psi(x)\theta(t)$, where $\Psi \in D$ and $\theta \in C_0^\infty(R^1)$.

REMARK. By virtue of Lemma 2.2, if $u(t) = W(t)f$, $f \in H$, then we may consider u as a distribution on $M \times R^1$. Thus we may speak of the support of u, supp $(u) \subseteq M \times R^1$, as well as of the support of $u(t)$, supp $(u(t)) \subseteq M$, for a single value of t. By Lemma 2.2 and a simple approximation argument, if $\mathcal{O} \subseteq M$ is open, then $u(t) = \mathcal{O}$ on 0 for $a < t < b$ is equivalent to $u = 0$ on $\mathcal{O} \times (a, b)$.

3. **Localization and domains of uniqueness.** If $W(t)=\exp itA$ is a one-parameter unitary group on H, and $f \in H$, let μ_f be the positive Borel measure on R^1 such that $(W(t)f, f)=\int \exp(it\lambda)\, d\mu_f(\lambda)$. We say that A has no homogeneous Lebesgue spectrum if none of the measures μ_f, $f \in H$, are equivalent (in the sense of mutual absolute continuity) to Lebesgue measure on R^1. For example, if the spectrum of A is a proper subset of R^1, then A has no homogeneous Lebesgue spectrum.

REMARK. This condition is clearly equivalent to the condition that the representation W of R^1 does not contain the regular representation.

THEOREM 3.1. *Let $K \subseteq M$, and let $u(t)=W(t)f$, $f \in H$. Suppose A has no homogeneous Lebesgue spectrum. Then* $\operatorname{supp}(u(t)) \subseteq K$ *for $t \leqq 0$ implies* $\operatorname{supp}(u(t)) \subseteq K$ *for all t.*

REMARK. Theorem 3.1 is also true if "$t \leqq 0$" is replaced by "$t \leqq t_0$" or "$t \geqq t_0$."

Proof. Let $E=\{f \in H \mid \operatorname{supp}(W(t)f) \subseteq K \text{ for } t \leqq 0\}$. Then E is a closed subspace of H. Indeed, E is the intersection of the null spaces of the continuous linear functionals $f \to \langle f, \Psi \rangle_1$, where $\Psi \in D_1$ and is supported on the complement of $\bar{K} \times (-\infty, 0]$ in M_1. Now E is obviously invariant under $W(t)$ for $t \leqq 0$. By Theorem 1 of [2], if A has no homogeneous Lebesgue spectrum then E must be invariant under $W(t)$ for all t. Thus if $f \in E$ and $t_0 \in R^1$, then $W(t_0)f \in E$, i.e., $\operatorname{supp}(W(t_0+t)f) \subseteq K$ for $t \leqq 0$. Hence $\operatorname{supp}(W(t_0)f) \subseteq K$. Q.E.D.

If $C \subseteq M \times R$ is open, we shall say that C is a *domain of uniqueness* for solutions u of equation (2.1) in case $\operatorname{supp}(u) \cap C = \text{void}$ implies $u=0$. If C is a domain of uniqueness, then by linearity two solutions of (2.1) which agree on C are equal everywhere. Obviously any domain of the form $M \times I$, I a nonempty open interval, is a domain of uniqueness. We establish next, using Theorem 3.1, that certain domains which are only asymptotically of this form for large negative t are also domains of uniqueness, under the continuing hypothesis that A have no homogeneous Lebesgue spectrum.

NOTATION. For $C \subseteq M \times R^1$, $t \in R^1$, let $C+t=\{(x, s+t) \mid (x, s) \in C\}$.

THEOREM 3.2. *Let $C \subseteq M \times R^1$ be open, such that $C+t \subseteq C$ for $t \leqq 0$ and $\bigcup_{t>0} C+t = M \times R^1$. If A has no homogeneous Lebesgue spectrum, then C is a domain of uniqueness for solutions of* (2.1).

REMARK. In case $M=R^N$, then any open cone in $R^{N+1}=M \times R^1$ containing the negative t-axis satisfies the conditions on C.

Proof. Let $u(t)=W(t)f$ and suppose $\operatorname{supp}(u) \cap C = \text{void}$. It suffices to show that $\operatorname{supp}(f) \cap \mathcal{O} = \text{void}$ for an arbitrary open set $\mathcal{O} \subseteq M$ with compact closure. Now $\bar{\mathcal{O}} \times \{0\} \subseteq \bigcup_{t>0} C+t$ and $C+t \subseteq C+s$ if $t<s$. Thus by compactness there exists a $t_0 \in R^1$ such that $\mathcal{O} \times \{0\} \subseteq C+t_0$. Hence for any $t \leqq 0$,

$$\mathcal{O} \times \{t\} \subseteq C+t_0+t \subseteq C+t_0,$$

so we conclude that $\mathcal{O} \times (-\infty, -t_0] \subseteq C$.

By hypothesis n vanishes on C, hence for $t \leqq -t_0$, supp $(u(t)) \subseteq \mathcal{O}'$ by the above. According to Theorem 3.1, this implies that supp $(u(t)) \subseteq \mathcal{O}'$ for all t. Hence $f = u(0)$ vanishes on $\mathcal{O}$. Q.E.D.

4. Some applications. Let $B \geqq 0$ be a self-adjoint operator on the Hilbert space $H = L_2(M)$ (Lebesgue measure on M). We shall establish sufficient conditions for the preceding results on localization and domains of uniqueness to be valid for certain spaces of solutions of the abstract "wave equation"

$$d^2u/dt^2 = -B^2u. \tag{4.1}$$

We assume that:

(4.2) B^{-1} exists as a bounded operator.

For α a real number let B^α be the nonnegative self-adjoint operator defined as usual via the spectral theorem. Define H_α to be the completion of $D[B^\alpha]$ with respect to the norm $\|f\|_\alpha = \|B^\alpha f\|_{L_2}$. (If $a \geqq 0$, $D[B^\alpha]$ is already complete in this norm, by (4.2).) Thus for $\alpha > \beta$, one has $H_\alpha \subset H_\beta$, the embedding arising from the inclusion $D[B^\alpha] \subseteq D[B^\beta]$, while $H_0 = H$.

Remark. In the case $B = (-\Delta + m^2)^{1/2}$, $m > 0$, Δ the Laplace operator on $L_2(R^N)$ (so that equation 4.1 is the Klein-Gordon equation), the spaces H_α are well-known spaces of tempered distributions. The cases $\alpha = 0$ and 1, or $\alpha = -\frac{1}{2}$ and $\frac{1}{2}$, are encountered in the energy norm, or the Lorentz invariant norm respectively, associated with the Klein-Gordon equation.

For $\alpha \geqq 0$ the elements of H_α are L_2 functions on M, and thus act as distributions in the usual way. For $\alpha < 0$, in order to realize H_α as a space of distributions we assume that:

(4.3_α) $D \equiv C_0^\infty(M) \subseteq D[B^{-\alpha}]$ and the map $B^{-\alpha}: D \to L_2(M)$ is continuous (relative to the topology of D and the norm topology of L_2) with dense range.

Lemma 4.1. *Suppose B satisfies* (4.2) *and* (4.3_α). *For every real number $\beta \geqq \alpha$, there exists a unique bilinear form on $H_\beta \times D$ satisfying conditions* (A), (B), (C) *of* §2, *such that for $f \in D[B^\beta]$ and $\phi \in D$, one has*

$$\langle f, \phi \rangle = \int_M f\phi \, dx. \tag{4.4}$$

Proof. As remarked above, the case $\alpha \geqq 0$ is trivial, the elements of H_β being locally integrable functions on M already. For α arbitrary, any such bilinear form is uniquely determined by (4.4), since $D[B^\beta]$ is dense in H_β.

Suppose $\alpha < 0$. Now by the spectral theorem the operator B is unitarily equivalent to a multiplication operator M_b on a space $L_2(\Omega)$, where b is a real-valued measurable function on Ω. Let this unitary equivalence be denoted by $f \to \hat{f}$; thus $\hat{D}$ is the image of D and $\hat{D}_\beta$ the image of $D[B^\beta]$.

By assumption (4.2), $b \geqq \lambda > 0$ for some constant λ. Obviously the map $f \to \hat{f}$ extends to an isomorphism of H_β with the space $\hat{H}_\beta$ of all measurable functions $\hat{g}$ on Ω such that $b^\beta \hat{g} \in L_2(\Omega)$. In particular, if $\hat{\phi} \in \hat{D}$, $\hat{g} \in \hat{H}_\beta$, and $\beta \geqq \alpha$, then $\hat{\phi}\hat{g} = b^{\alpha-\beta} b^{-\alpha}\hat{\phi} b^\beta \hat{g} \in L_1(\Omega)$, since $b^{\alpha-\beta} \in L_\infty$ and $b^{-\alpha}\hat{\phi} \in L_2$. Define

$$\langle g, \phi \rangle = \int_\Omega \hat{g}(\phi^*)^{\wedge *}\, dw$$

(the asterisk denoting complex conjugation).

Clearly if $g \in L_2(M)$, then $\langle g, \phi \rangle = (g, \phi^*)_{L_2} = \int g\phi\, dx$. Furthermore, if a sequence $\phi_n \to 0$ as elements of D, then so does the sequence ϕ_n^*, and by (4.2) $B^{-\alpha}\phi_n^* \to 0$ in $L_2(M)$. Hence for $\beta \geqq \alpha$, $b^{-\beta}\hat{\phi}^* \to 0$ in $L_2(\Omega)$ and consequently $\langle g, \phi_n \rangle \to 0$ for every $g \in H_\beta$. Also $|\langle g, \phi \rangle| \leqq \|b^{\beta-\alpha}\|_\infty \|g\|_\beta \|b^{-\alpha}\hat{\phi}_n^*\|_{L_2}$, hence $g \to \langle g, \phi \rangle$ is continuous. Finally, if $g \in H_\beta$ and $\langle g, \phi \rangle = 0$ for all $\phi \in D$, then $b^\alpha \hat{g} \perp b^{-\alpha}\hat{D}$ in $L_2(\Omega)$. But by assumption, $b^{-\alpha}\hat{D}$ is dense in $L_2(\Omega)$, so $b^\alpha \hat{g} = 0$ a.e. Since $b^\alpha > 0$ a.e., it follows that $g = 0$. This establishes the existence of a bilinear form with the requisite properties. Q.E.D.

For examples of operators satisfying conditions (4.2) and (4.3_α), we mention first the operator $B = (-\Delta + m^2)^{1/2}$, $m > 0$, in its usual self-adjoint formulation, on $L_2(R^N)$. $B \geqq m$, and is hence invertible. Condition (4.3_α) is satisfied for all $\alpha < 0$. Indeed, the dense range condition follows from the essential self-adjointness of $B^{-\alpha}$ on the translation-invariant domain D (cf. [6]) and the invertibility of $B^{-\alpha}$; the other conditions are easily verified. Further examples arise from perturbations of the above. Suppose that V is a nonnegative operator on D which is of "smaller order" than B^2, i.e., there exist constants a, b with $0 \leqq a < 1$ such that

$$\|V\phi\| \leqq a\|B^2\phi\| + b\|\phi\|, \tag{4.5}$$

for $\phi \in D$. As is well known from the work of Kato and others, this implies that $A = B^2 + V$ is essentially self-adjoint on D (since B^2 is ess. s.a. on D and V has a closure, it follows that inequality (4.5) extends to all ϕ in $D[B^2]$). Hence the conditions (4.2) and (4.3_α) are satisfied for $B_1 = A^{-1/2}$, with $\alpha = -2$, in which case equation (4.1) becomes $d^2u/dt^2 = -(B^2 + V)u$.

We now return to equation (4.1), and show using the spaces H_α that it can be brought to the form (2.1), by the familiar reduction of higher-order equations to systems of first-order equations.

Let $T_\alpha = B^2 : H_{\alpha+1} \to H_\alpha$, with $D[T_\alpha] = H_{\alpha+2} \subseteq H_{\alpha+1}$ (i.e., in the notation of Lemma 4.1, $D[T_\alpha]^\wedge = \{\hat{f}$ measurable on $\Omega \mid b^{\alpha+2}\hat{f} \in L_2(\Omega)\}$, and $\hat{T}_\alpha \hat{f} = b^2\hat{f}$.). Then T_α^* is the identity map from $H_{\alpha+1}$ (with norm $\|\cdot\|_\alpha$) to $H_{\alpha+1}$ (with norm $\|\cdot\|_{\alpha+1}$). Thus equation (4.1) may be written as

$$\begin{aligned} du/dt &= iT_\alpha^* v, \qquad u(0) \in H_{\alpha+1}, \\ dv/dt &= iT_\alpha u, \qquad v(0) \in H_\alpha. \end{aligned} \tag{4.6}$$

(Just as in the case of equation (2.1), this only makes literal sense when $u(t) \in H_{\alpha+2}$ and $v(t) \in H_{\alpha+1}$. We are only interested in the integrated form of the equations, however.)

Let $K_\alpha = H_{\alpha+1} \oplus H_\alpha$. Take A to be the self-adjoint operator on K_α with $D[A] = D[T_\alpha] \oplus D[T_\alpha^*]$ and matrix

$$\begin{pmatrix} 0 & T_\alpha^* \\ T_\alpha & 0 \end{pmatrix}$$

relative to the given direct sum decomposition of K_α. Equation (4.6) then assumes the form (2.1).

THEOREM 4.1. *The conclusions of Theorems* 3.1 *and* 3.2 *hold for weak solutions* $u(t)$ *of* (4.1), *provided* $u(0) \in H_{\alpha+1}$, $\dot{u}(0) \in H_\alpha$ *for some real* α, *and* B *satisfies conditions* (4.2) *and* (4.3_α).

REMARK. To be precise, $u(t)$ is assumed to be the first component of a pair $u(t) \oplus v(t) \in K_\alpha$, where

$$u(t) \oplus v(t) = e^{itA}(u_0 \oplus v_0), \qquad u_0 = u(0), \qquad v_0 = \dot{u}(0), \tag{4.7}$$

and the elements of $H_{\alpha+1}$, H_α act as distributions via Lemma 4.1. The dot denotes differentiation with respect to t (cf. Lemma 4.2). We may assume $\alpha \leqq 0$.

Proof of theorem. By Assumption (4.2), there exists $\varepsilon > 0$ such that $B \geqq \varepsilon$. A direct calculation shows that A is unitarily equivalent to the operator

$$\begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}$$

on K_α with domain $H_{\alpha+2} \oplus H_{\alpha+1}$, and hence the spectrum of A excludes the interval $(-\varepsilon^{\alpha+1/2}, \varepsilon^{\alpha+3/2})$. In particular, A has no homogeneous Lebesgue spectrum. It only remains, therefore, to show that if $\operatorname{supp}(u(t)) \subseteq K$ for $a \leqq t \leqq b$, then the same is true for $v(t)$, where u and v are related by (4.7). This follows from the following lemma:

LEMMA 4.2. *Let u and v be related by equation* (4.7). *Then for every* $\phi \in D$,

$$d/dt\langle u(t), \phi\rangle = \langle v(t), \phi\rangle \tag{4.8}$$

Proof. We may express $u(t)$ and $v(t)$ in terms of the initial data u_0 and v_0 by the familiar formulas

$$u(t) = \cos tBu_0 + B^{-1} \sin tBv_0, \qquad v(t) = -B \sin tBu_0 + \cos tBv_0$$

(see [8], for example). Let $\phi \in D$. Then using the notation of Lemma 4.1, we note that $\hat{\phi} b \hat{u}_0 \in L_1$, and $(d/dt)\hat{\phi} \cos tb\hat{u}_0 = -\hat{\phi} b \sin tb\hat{u}_0$, pointwise on Ω, with the difference quotients bounded by $3\hat{\phi} b \hat{u}_0$. Hence by the dominated convergence theorem, $(d/dt) \int \hat{\phi} \cos tb\hat{u}_0 \, dw = -\int \hat{\phi} b \sin tb\hat{u}_0 \, dw$. A similar argument applies for the term $B^{-1} \sin tBu_1$, and adding yields (4.8).

5. **Half-cylinders as domains of uniqueness.** In special cases, Theorem 3.1 together with standard arguments (cf. [3, Theorem 7]) can be used to obtain results on domains of uniqueness which do not require the t-sections of the domain to grow as $t \to -\infty$, in contrast to the domains considered in Theorem 3.1. For simplicity we shall only consider the Klein-Gordon equation. Let $B = (-\Delta + m^2)^{1/2}$, $m > 0$, acting on $L_2(R^N)$, and define the spaces H_α associated with B as in §4.

THEOREM 5.1. *Let u be a weak solution of $\Box u = m^2 u$, $m \neq 0$, such that for some real α, $u(0) \in H_{\alpha+1}$, $\dot{u}(0) \in H_\alpha$. Suppose $\mathcal{O} \subseteq R^N$ is a nonempty open set such that $u(t)$ vanishes on $\mathcal{O}$ for $t \leqq 0$. Then $u = 0$.*

REMARKS. $\Box$ denotes the operator $\Delta - \partial^2/\partial t^2$. The sense in which u is a solution of the equation is that of Theorem 4.1.

Proof. By translation invariance, we may assume $\mathcal{O}$ is a neighborhood of 0 in R^N. By the results of §§3 and 4, $u(t)$ must vanish on $\mathcal{O}$ for all t.

Now $u(t)$ is a tempered distribution with Fourier transform

$$\cos tb\hat{u}_0 + b^{-1} \sin tb\hat{u}_1,$$

where $b(\xi) = (|\xi|^2 + m^2)^{1/2}$, $\xi \in R^N$, $u_0 = u(0)$, $u_1 = \dot{u}(0)$, and $b^{\alpha+1}\hat{u}_0 \in L_2$, $B^\alpha \hat{u}_1 \in L_2$. By the vanishing of $u(t)$ on $\mathcal{O}$, if $\phi \in C_0^\infty(\mathcal{O})$, then

$$\int \hat{\phi}[\cos tb\hat{u}_0 + b^{-1} \sin tb\hat{u}_1]\, d\xi = 0$$

for all t. Let $f_\pm = \frac{1}{2}[\hat{u}_0 \mp ib^{-1}\hat{u}_1]$. Then $b^{\alpha+1} f_\pm \in L_2$, and

$$\cos tb\hat{u}_0 + b^{-1} \sin tb\hat{u}_1 = e^{itb} f_+ + e^{-itb} f_-.$$

Hence

$$\int e^{itb}\hat{\phi} f_+ = -\int e^{-itb}\hat{\phi} f_-. \tag{5.1}$$

Since $b \geqq 0$, the left-hand side of (5.1) is the boundary value of a function bounded and holomorphic for Im $t > 0$, while the right-hand side is the boundary value of a function bounded and holomorphic for Im $t < 0$. The equality for real t thus implies by analytic continuation that each side is constant. Since $|e^{itb}| \leqq e^{-ms}$, $s = \text{Im}(t)$, that constant must be zero (let Im $t \to \infty$).

Thus we may deal with $f_\pm$ separately, and it clearly suffices to show $f_\pm = 0$. (Up to this point we have only used the positivity of the operator B.) Consider $f = f_+$ (the argument for f_- is the same). Since $b^{\alpha+1} f \in L_2$, f is locally integrable. Let S^{N-1} be the unit sphere $\{|\xi| = 1\}$ in R^N, and $d\sigma$ the invariant measure on S^{N-1}. For $\phi \in C_0^\infty(\mathcal{O})$, set $g(r) = \int_{S^{N-1}} \hat{\phi}(r\sigma) f(r\sigma)\, d\sigma$, $r > 0$. By Fubini's theorem and the above argument, $\int_0^\infty \exp itb(r) g(r) r^{N-1}\, dr = 0$ for all t. By the change of variable $\lambda = b(r)$ this implies $\int_m^\infty e^{it\lambda}\, d\mu(\lambda) = 0$, where

$$d\mu(\lambda) = g(r(\lambda))\lambda(\lambda^2 - m^2)^{N-3/2}\, d\lambda.$$

But μ is a finite measure, and by the uniqueness of Fourier-Stieltjes transforms, $\mu = 0$, hence $g = 0$ a.e.

Now $C_0^\infty(\mathcal{O})$ is invariant under differentiation, hence taking Fourier transforms, we conclude that for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$ and almost all r,

$$\int_{S^{N-1}} \sigma^\alpha \hat{\phi}(r\sigma) f(r\sigma)\, d\sigma = 0$$

($\sigma^\alpha = \sigma_1^{\alpha_1} \cdots \sigma_N^{\alpha_N}$, where $\sigma_1, \ldots, \sigma_N$ are the rectangular coordinates of $\sigma \in S^{N-1}$). Since $\hat{\phi} f \in L_1(d\sigma)$, we conclude that for almost all r, $\hat{\phi}(r\sigma) f(r\sigma) = 0$ for almost all σ. Applying Fubini's theorem, we conclude that $\hat{\phi} f = 0$ a.e. on R^N. Choosing a sequence $\phi_n \in C_0^\infty(\mathcal{O})$ such that $\hat{\phi}_n \to 1$ we obtain $f = 0$ a.e. as desired. Q.E.D.

REMARK. Theorem 5.1 may also be proved by appealing first to the general theorem that u must vanish in the full backward light cone [9, Theorem 5.3.3], and then applying Theorem 4.1 of the present paper.

6. **A counter-example.** The converse to Theorem 3.2, namely the presence of homogeneous Lebesgue spectrum implying the existence of nonzero solutions of (2.1) vanishing on C, depends on the pairing $\langle \cdot, \cdot \rangle$, i.e., the notion of localization employed. For an example where the converse does hold (cf. [3]), where nonzero finite-energy solutions to the scalar wave equation which vanish on the backward light cone are constructed. Here we give an example where it does not hold.

Let $H = L_2(R^N)$, and let T be a bounded antilocal self-adjoint translation-invariant operator on H (e.g., $T = (I - \Delta)^\lambda$ for $\lambda < 0$, $\lambda \neq$ integer, N odd (see [7])). Here *antilocal* means that whenever f and Tf both vanish on a nonempty open set, then $f = 0$. Let $D = C_0^\infty(R^N) \oplus C_0^\infty(R^N)$, and define a pairing on $H \times D$ by

$$\langle f, \phi_1 \oplus \phi_2 \rangle = \int \phi_1 Tf\, dx + \int \phi_2 f\, dx. \tag{6.1}$$

Clearly this pairing satisfies conditions (A), (B), and (C) of §2.

Let $A = \partial / i\, \partial x_1$ on H, so that $W(t) = e^{itA}$ acts by translation of x_1:

$$W(t) f(x_1, \ldots, x_N) = f(x_1 + t, \ldots, x_N).$$

Thus W is a multiple of the regular representation of R^1 with multiplicity $\aleph_0$. However, if $f \in H$, $f \neq 0$, then $u(t) = W(t)f$, considered as an element of D' via the pairing (6.1), has global support for all t, by the antilocality of T.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASSACHUSETTS